(c) Using the Frisch-Waugh-Lovell theorem, least squares on the original model yield

$$\hat{\beta}_2 = (x_2'\bar{P}_{X_1}x_2)^{-1}x_2'\bar{P}_{X_1}y = (x_2'x_2 - x_2'P_{X_1}x_2)^{-1}x_2'\bar{P}_{X_1}y$$
$$= (1-a)^{-1}(x_2'x_2)^{-1}x_2'\bar{P}_{X_1}y = b_2^{(1)}/(1-a).$$

As $j \to \infty$, $\lim b_2^{(j+1)} \to (\sum_{j=0}^{\infty} a^j) b_2^{(1)} = b_2^{(1)}/(1-a) = \hat{\beta}_2$.

NOTE

1. An excellent solution has also been proposed by Denzil G. Fiebig, the poser of the problem.

95.5.2. *The Null Distribution of Nonnested Tests with Nearly Orthogonal Regression Models*—Solution, proposed by Leo Michelis.

(a) Condition (A3) implies that the asymptotic conditional covariance, Σ_{xz}, between X and Z given W is zero, so that the nonnested models are asymptotically orthogonal. Because Δ is $O(1)$ by assumption, asymptotic orthogonality is attained at a rate proportional to $n^{-1/2}$.

(b)[1] The J test is the t statistic for $\alpha = 0$ in the artificial regression

$$y = \tilde{X}\beta + \alpha P_{\tilde{Z}}y + \varepsilon, \tag{1}$$

where $\tilde{X} = M_W X$, $\tilde{Z} = M_W Z$, and $P_{\tilde{Z}} = \tilde{Z}(\tilde{Z}'\tilde{Z})^{-1}\tilde{Z}'$. The simplified Cox ($C$) test is

$$C = n(\hat{\omega}^2 - \hat{\omega}_*^2)/\sqrt{\widehat{\text{var}}[n(\hat{\omega}^2 - \hat{\omega}_*^2)]}, \tag{2}$$

where $\hat{\omega}^2$ and $\hat{\omega}_*^2$ are the ML estimates of ω^2 and ω_*^2, the pseudo-true value of ω^2 under H_0, respectively. Under H_0, it is easily shown that asymptotically the J and $-C$ statistics can be written as

$$J \overset{a}{=} \frac{\beta'\tilde{X}P_{\tilde{Z}}u + u'P_{\tilde{Z}}u}{\sigma(\beta'\tilde{X}P_{\tilde{Z}}\tilde{X}\beta + 2\beta'\tilde{X}P_{\tilde{Z}}u + u'P_{\tilde{Z}}u)^{1/2}} + o_p(1) \tag{3}$$

$$-C \overset{a}{=} \frac{2\beta'\tilde{X}P_{\tilde{Z}}u + uP_{\tilde{Z}}u}{2\sigma(\beta'\tilde{X}P_{\tilde{Z}}\tilde{X}\beta)^{1/2}} + o_p(1), \tag{4}$$

where in view (A1), (A2), and (A3) each quantity in (3) and (4) is $O_p(1)$. Let k be a scalar. Then, asymptotically under (A2) and (A3), it is easily seen that

$$n^{1/2}\tilde{Z}'(u + k\tilde{X}\beta) \sim N(k\Delta'\beta, \sigma^2\Sigma_{zz}). \tag{5}$$

Now let $\chi_q^2(\lambda)$ denote a χ^2 variate with q degrees of freedom and noncentrality parameter λ and also let

$$R(k) \equiv \sigma^{-2}(u + k\tilde{X}\beta)P_{\tilde{Z}}(u + k\tilde{X}\beta). \tag{6}$$

Then, given (5) and $P_{\tilde{Z}} = (n^{-1/2}\tilde{Z})(n^{-1}\tilde{Z}'\tilde{Z})^{-1}(n^{-1/2}\tilde{Z}')$, we have

$$R(k) = \chi_q^2(k^2c^2) + o_p(1), \tag{7}$$

where $c^2 = \sigma^{-2}\beta'\Delta\Sigma_{22}^{-1}\Delta'\beta$. Note that (6) can also be written as

$$R(k) = \sigma^{-2}uP_{\tilde{Z}}u + \sigma^{-2}2k\beta'\tilde{X}P_{\tilde{Z}}u + k^2c_n^2, \tag{8}$$

where $c_n^2 = \sigma^{-2}\beta'\tilde{X}P_{\tilde{Z}}\tilde{X}\beta$. Considering the J statistic, (8) can be used to rewrite (3) as $\{R(\frac{1}{2}) - c_n^2/4\}^{1/2}/\{R(1)\}^{1/2}$ which, given (7), shows that asymptotically

$$J \overset{a}{=} \frac{4\chi_q^2(c^2/4) - c^2}{4\{\chi_q^2(c^2)\}^{1/2}} + o_p(1) \tag{9}$$

as $\text{plim}(c_n^2) = c^2$. Similarly for the Cox statistic, (4) can be written as $\{R(1) - c_n^2\}/2c_n$, so (7) shows that asymptotically

$$-C \overset{a}{=} \frac{\chi_q^2(c^2) - c}{2c} + o_p(1). \tag{10}$$

(c) When $X'M_WZ = 0$, then $c = 0$. Consequently, asymptotically, (9) shows that $J^2 \sim \chi_q^2(0)$ and (10) shows that the Cox statistic is undefined. The numerator of (10), however, is also distributed as $\chi_q^2(0)$.

NOTE

1. In answering (b), I have benefited from a suggestion made by an anonymous referee.

95.5.3. *The Moore–Penrose Inverse of a Sum of Three Matrices*—Solution,[1] proposed by Heinz Neudecker.

(i) Write $V = aI + bZZ^+ + cXX^+$. Consider then

$$W = \alpha I + \beta ZZ^+ + \gamma XX^+.$$

Multiplication of V and W yields

$$VW = \alpha a I + (\alpha b + \beta(a+b))ZZ^+ + ((\alpha + \beta)c + \gamma(a+b+c))XX^+,$$

by virtue of the identity

$$ZZ^+XX^+ = XX^+,$$

which follows from $X = ZJ$, J being a selection matrix. Imposition of $\alpha a = 1$, $\alpha b + \beta(a+b) = (\alpha+\beta)c + \gamma(a+b+c) = 0$ yields

$$V^{-1} = W = a^{-1}I - a^{-1}(a+b)^{-1}bZZ^+ - (a+b)^{-1}(a+b+c)^{-1}cXX^+.$$

We have to require that $a \neq 0$, $a + b \neq 0$, and $a + b + c \neq 0$. Clearly, $a > 0$, $a + b > 0$, and $a + b + c > 0$ is sufficient.

(ii) $V = aI + bZZ^+ + cXX^+$.

Because ZZ^+ and XX^+ are symmetric and commutative, they can be diagonalized by an orthogonal matrix, say $\mathbf{T}$.

Thus,

$$ZZ^+ = T\Lambda T', \quad XX^+ = TMT'.$$