Large Distributional Games with Traits*

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Abstract: A comprehensive theory of large strategic games with (socioeconomic and biological) traits (*LSGT*) has recently been presented in Khan et al. (2012 a and b), and in this paper, we present a reformulation pertaining to large distributional games with traits (*LDGT*). In addition to a generalization of work initiated and advocated by Mas-Colell (1984), we delineate the role of saturated spaces, as studied in Keisler-Sun (2009), in the reformulated theory, and consider questions pertaining to “realizations” of equilibrium distributions that were not previously asked. (84 words)

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1 Introduction

The theory of large non-atomic games in strategic form has by now gone well beyond Schmeidler's (1973) framework in which a player's choice among a finite number of actions depends on a statistical summary, be it an average or a distribution, of the plays of everyone else in the game. Recent work of Khan et al. (2012a and b) has offered a rich but analytically tractable framework for game-theoretic situations in which the payoffs of the non-denumerable number of players depend not only on the actions chosen by other players but also on underlying characteristics of the players themselves, be they socioeconomic or biological.¹ The resulting theory does not conflate *names* and *traits*, and with these separated out, goes beyond existence of pure-strategy equilibria to obtain comprehensive results on the continuity, asymptotic implementation and *ex-post* characterizations of such equilibria. However, the theory requires the jettisoning of the Lebesgue interval as the space of player-names in favor of so-called saturated measure spaces; indeed it offers a characterization of such spaces.

A decade subsequent to Schmeidler's paper, Mas-Colell (1984) proposed the jettisoning of the space of player-names altogether, and presented a "reformatization ... in terms of distributions rather than measurable functions." He emphasized the simplicity of his conception.

We shall see how once the definitions are available we get a (pure strategy) equilibrium existence theorem quite effortlessly and under general conditions ... Because of (insubstantial) measurability problems an a priori given representation of the game is thus a sort of straightjacket. The approach via distributions frees one from it. This accounts for its comparative ease.

Indeed, if one invokes the identity map and sees a player's name also as his characteristic, it is now well-understood that in a setting with finite actions, Mas-Colell's theorem is an almost trivial consequence of that of Schmeidler, and it is only in more general contexts that the process of *symmetrization*, as opposed to *purification*, can be exploited, and connections with the "marriage lemma" and "disintegrations" made, resulting in the distributional approach coming into its own.² In any case, the variant has attracted both theoretical and applied interest, and the two approaches, one based on a random variable and the other on its law, are simply complementary ways of looking at a large game, two sides of the same coin.

¹These considerations have been emphasized in Akerlof-Krampton (2000, 2002) and Brock-Durlauf (2001); see Blume et al. (2010) and their references.

²See Khan-Sun (1994, 1995) for these claims. For general statements of the theory, see Khan (1989), Khan-Sun (1990) and Rath (1996).

With this view of the antecedent background literature, the following questions naturally suggest themselves:

- (i) What shape does the theory of large strategic games with traits (henceforth *LSGT*), with its informationally-rich notion of agent interdependence, take when recast for large distributional games with traits (henceforth *LDGT*)?
- (ii) What role does the notion of a saturated probability space play in this reformulated theory?

We answer each of these questions here in a way that has the satisfying feature that it recovers previous results simply by the specialization of the separable metric space of traits to a singleton! And to be sure, writing two and half decades after Mas-Colell, we can rely on previous advances and bring to bear heavier mathematical artillery to resolve questions that he did not address. All in all, the results bear testimony not only to the simplicity but the fecundity of Mas-Colell's conception.

This letter is then organized as follows. Section 1 introduces saturated spaces and uses them to record an antecedent result on *LSGT*, Proposition 1. Section 2 turns to *LDGT* and presents the necessary definitions of the game, the Nash and symmetric Nash equilibrium distributions of the game, *NED* and symmetric *NED*, the realization of these distributions and their closed graph property. Section 3 presents the results. Theorem 1 and 2 generalize the corresponding results on *NED* in Mas-Colell (1984) and Khan-Sun (1995). The other results represent the extended reach of the theory: Theorem 3 is on the closed graph property of a *NED* correspondence; Proposition 2, Theorem 4 and Corollary 1 are results on Lebesgue and saturated realizations of *NED*, symmetric and non-symmetric. The proofs of these results are all in keeping with Mas-Colell's original conception – they are natural and effortless consequences of Keisler-Sun (2009) and Khan et al. (2012a) once the necessary definitions are in place.

2 Large Strategic Games with Traits (*LSGT*)

The principal motivation behind the development of *LSGT* is the need for a rich space of player characteristics, embodied in a universal space of traits T , a complete separable metrizable (Polish) space, and a particular distribution ρ on the induced Borel σ -algebra $\mathcal{B}(T)$ on this space. The resulting “externality” notion then embraces both distributions on T and the common action set A assumed to be compact metric. This involves $\mathcal{M}(T \times A)$,

the space of probability distributions on the Borel product σ -algebra on $T \times A$, and its subspace $\mathcal{M}^\rho(T \times A)$ of distributions whose marginal distribution on T is ρ . The space of players' payoffs $\mathcal{V}_{(A,T,\rho)}$ is then given by the space of all continuous functions on the product space $A \times \mathcal{M}^\rho(T \times A)$, and endowed with its resulting Borel σ -algebra induced by the sup-norm topology. It is the measurable space $(\mathcal{V}_{(A,T,\rho)}, \mathcal{B}(\mathcal{V}_{(A,T,\rho)}))$. The space of players' names $(I, \mathcal{I}, \lambda)$, an atomless probability space, thus does not figure as part of the players' characteristics. We can now present a formulation of a *LSGT* as follows.

Definition 1. A *LSGT* is a measurable function $\mathcal{G}$ from I to $T \times \mathcal{V}_{(A,T,\rho)}$ such that $\lambda \mathcal{G}_1^{-1} = \rho$, where $\mathcal{G}_k$ is the projection of $\mathcal{G}$ on its k^{th} -coordinate, $k = 1, 2$. A (pure strategy) Nash equilibrium of $\mathcal{G}$ is a measurable function $f : I \rightarrow A$, such that for λ -almost all $i \in I$, and with v_i abbreviated for $\mathcal{G}_2(i)$, and $\alpha : I \rightarrow T$ abbreviated for $\mathcal{G}_1$,

$$v_i(f(i), \lambda(\alpha, f)^{-1}) \geq v_i(a, \lambda(\alpha, f)^{-1}) \text{ for all } a \in A.$$

In order to develop a general result on the existence of a Nash equilibria in *LSGT*, we need the notion of a saturated space as a formalization of the space of player-names.

Definition 2. A probability space is said to be countably-generated if its σ -algebra can be generated by a countable number of subsets together with the null sets; otherwise, it is not countably-generated. A probability space $(I, \mathcal{I}, \lambda)$ is saturated if it is nowhere countably-generated, in the sense that, for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ is not countably-generated, where $\mathcal{I}^S := \{S \cap S' : S' \in \mathcal{I}\}$ and λ^S is the probability measure re-scaled from the restriction of λ to $\mathcal{I}^S$.

From the above definition, it is clear that saturated probability spaces must be atomless. An important property of saturated spaces is with respect to the saturation property defined below and encapsulated in a fact due to Hoover-Keisler and available in Keisler-Sun (2009).

Definition 3. An atomless probability space $(I, \mathcal{I}, \lambda)$ is said to have the saturation property for a Borel probability measure ν on the product of Polish spaces $X \times Y$ if for every measurable mapping $f : I \rightarrow X$ which induces the distribution of the marginal measure of ν over X , then there is a measurable mapping $g : I \rightarrow Y$ such that the induced distribution of the pair (f, g) on $(I, \mathcal{I}, \lambda)$ is ν .

Fact 1. A probability space $(I, \mathcal{I}, \lambda)$ is saturated if and only if it has the saturation property for every Borel probability measure ν on the product of any two Polish spaces.

The following antecedent result is from Khan et al. (2012a, Theorem 1).

Proposition 1. *Every LSGT $\mathcal{G} : I \longrightarrow T \times \mathcal{V}_{(A,T,\rho)}$ has a Nash equilibrium if either of the following two (sufficient) conditions hold: (i) T and A are both countable spaces, (ii) $(I, \mathcal{I}, \lambda)$ is a saturated probability space.*

3 Large Distributional Games with Traits (LDGT)

We now turn to the formulation where the space of player-names is suppressed, and rather than measurable functions, the focus shifts to their distributions.³

Definition 4. *A LDGT is a distribution μ^ρ on $T \times \mathcal{V}_{(A,T,\rho)}$ such that μ_T^ρ , the marginal of μ^ρ on T , is ρ . A NED of μ^ρ is a distribution τ on $T \times \mathcal{V}_{(A,T,\rho)} \times A$ if the marginal of τ on $T \times \mathcal{V}_{(A,T,\rho)}$ is μ^ρ , and if $\tau(B_\tau) = 1$ where*

$$B_\tau = \{(t, v, a) \in (T \times \mathcal{V}_{(A,T,\rho)} \times A) : v(a, \tau_{T \times A}) \geq v(x, \tau_{T \times A}) \text{ for all } x \in A\},$$

and $\tau_{T \times A}$ is the marginal of τ on $T \times A$.

It is of interest to examine if players with identical characteristics (identical trait and identical payoff function) take an identical action in a NED.

Definition 5. *A NED τ^s of a LDGT μ^ρ is symmetric if there exists a measurable function $h : T \times \mathcal{V}_{(A,T,\rho)} \longrightarrow A$ such that $\tau^s(\text{graph of } h) = 1$. A NED τ of μ^ρ can be symmetrized if there exists a symmetric NED τ^s of μ^ρ such that $\tau_{T \times A}^s = \tau_{T \times A}$.*

Next, we offer a definition of the closed graph property of the NED correspondence.

Definition 6. *The NED correspondence of a LDGT μ^ρ has the closed graph property if for any sequence of LDGT $\{\mu^n\}$ which converges weakly to μ^ρ in $\mathcal{M}(T \times \mathcal{V}_{(A,T,\rho)})$ and any sequence $\{\tau^n\}$ which converges weakly to some $\tau \in \mathcal{M}(T \times \mathcal{V}_{(A,T,\rho)} \times A)$, where for all n the marginal of each μ^n on T is ρ and τ^n is a NED of μ^n , then τ is a NED of μ^ρ .*

Finally, we present the notion of a realization of a NED, studied in Keisler-Sun (2009). In addition to it being of interest in its own right, it plays an essential role in the proofs.

Definition 7. *A probability space $(I, \mathcal{I}, \lambda)$ realizes a NED τ of a LDGT μ^ρ if every LSGT $\mathcal{G} : I \longrightarrow T \times \mathcal{V}_{(A,T,\rho)}$ satisfying $\lambda \mathcal{G}^{-1} = \mu^\rho$ has a Nash equilibrium f such that $\lambda(\mathcal{G}, f)^{-1} = \tau$.*

³In previous work, this resulting game form LDGT was referred to as a “large anonymous game,” and its equilibrium distribution referred to as a CNED.

4 The Results and their Proofs

We first provide the existence result of a *NED* in a *LDGT*. It is worth emphasizing that (like Proposition 1 (ii)), we do not need to impose any cardinality requirement on T or A for this existence result.

Theorem 1. *There exists a NED for any LDGT.*

Proof: Let μ^ρ be an *LDGT* and $(I, \mathcal{I}, \lambda)$ be a saturated probability space. As $(I, \mathcal{I}, \lambda)$ must be atomless, it is known that there exists an $\mathcal{I}$ -measurable mapping $\mathcal{G} : I \rightarrow T \times \mathcal{V}_{(A, T, \rho)}$ such that $\lambda \mathcal{G}^{-1} = \mu^\rho$, see e.g., Keisler-Sun (2009, Lemma 2.1). Clearly, $\mathcal{G}$ is a LSGT. Therefore, by Proposition 1 (ii), there exists a Nash equilibrium f of $\mathcal{G}$. Let $\tau = \lambda(\mathcal{G}, f)^{-1}$. It is easy to show that τ is a *NED* of μ^ρ . ■

The next result shows that symmetric equilibrium distributions exist when T and A are countable and μ^ρ is atomless. Moreover, under the same conditions, any equilibrium distribution can be symmetrized.

Theorem 2. *There exists a symmetric NED of an atomless LDGT if T and A are countable. Furthermore, every NED of such a LDGT can be symmetrized.*

Proof: Let μ^ρ be a *LDGT*. By Theorem 1, it has a *NED*. Let τ be an arbitrary *NED* of μ^ρ . Thus to prove this theorem, it is sufficient to show that τ can be symmetrized. For any $t \in T$ and $a \in A$, let $W_{t,a} = \{(t, v) \in T \times \mathcal{V}_{(A, T, \rho)} : (t, v, a) \in B_\tau\}$. Since both T and A are countable, $\{W_{t,a}\}_{t \in T, a \in A}$ is a countable family of subsets of $\mathcal{B}(T \times \mathcal{V}_{(A, T, \rho)})$. For any finite subset D_F of $T \times A$,

$$\begin{aligned} \mu^\rho \left(\bigcup_{(t,a) \in D_F} W_{t,a} \right) &= \tau \left(\left(\bigcup_{(t,a) \in D_F} W_{t,a} \right) \times A \right) \geq \sum_{(t,a) \in D_F} \tau(W_{t,a} \times \{a\}) \\ &= \sum_{(t,a) \in D_F} \tau(\{t\} \times \mathcal{V}_{(A, T, \rho)} \times \{a\}) = \sum_{(t,a) \in D_F} \tau_{T \times A}(\{t\} \times \{a\}). \end{aligned}$$

As μ^ρ is atomless, by Bollobás and Varopoulos's marriage lemma, as used in Khan-Sun (1995), there exists a disjoint family $\{V_{t,a}\}_{(t,a) \in T \times A}$ of sets in $\mathcal{B}(T \times \mathcal{V}_{(A, T, \rho)})$ such that $V_{t,a} \subseteq W_{t,a}$, $\mu^\rho(V_{t,a}) = \tau_{T \times A}(\{t\} \times \{a\})$. Now define $g : T \times \mathcal{V}_{(A, T, \rho)} \rightarrow T \times A$ such that for all $a \in A$ and for almost all $(t, v) \in V_{t,a}$, $g(t, v) = a$. It is easy to check that the measure $\mu^\rho(i_d, g)^{-1}$ is the required symmetrization where i_d is the identity mapping on $T \times \mathcal{V}_{(A, T, \rho)}$. ■

For any *LDGT* μ^ρ , consider any *LSGT* $\mathcal{G}$ with a saturated space $(I, \mathcal{I}, \lambda)$ as its name space such that $\lambda\mathcal{G}^{-1} = \mu^\rho$. For any given *NED* of μ^ρ , By fact 1, there exists a pure strategy profile f of $\mathcal{G}$ such that $\lambda(\mathcal{G}, f)^{-1} = \tau$. It is clear that f is a Nash equilibrium of $\mathcal{G}$. Thus, we have the following result.

Proposition 2. *A NED of any LDGT can be realized by any saturated probability space.*

With this result, we can show the closed graph property of the *NED* correspondence.

Theorem 3. *The NED correspondence of any LDGT has the closed graph property.*

Proof: Fix a *LDGT* μ^ρ and a saturated probability space $(I, \mathcal{I}, \lambda)$. Let $\{\mu^n\}$ converges weakly to μ^ρ in $\mathcal{M}(T \times \mathcal{V}_{(A, T, \rho)})$ and $\{\tau^n\}$ converge weakly to some $\tau \in \mathcal{M}(T \times \mathcal{V}_{(A, T, \rho)} \times A)$, where the marginal of μ^n on T is ρ and τ^n is the *NED* of μ^n for all n . As $(I, \mathcal{I}, \lambda)$ is atomless, there exists a *LSGT* $\mathcal{G} : I \longrightarrow T \times \mathcal{V}_{(A, T, \rho)}$ such that $\lambda\mathcal{G}^{-1} = \mu$, and there exists a *LSGT* $\mathcal{G}^n : I \longrightarrow T \times \mathcal{V}_{(A, T, \rho)}$ where $\lambda(\mathcal{G}^n)^{-1} = \mu^n$ for each n . Furthermore, by Proposition 2, for a *NED* τ^n , there exists a Nash equilibrium f^n of $\mathcal{G}^n$ such that $\lambda(\mathcal{G}^n, f^n)^{-1} = \tau^n$ for all n . By Khan et al. (2012a, Theorem 4) and its proof, there is a Nash equilibrium f of $\mathcal{G}$ such that $\lambda(\mathcal{G}, f)^{-1} = \tau$. It is clear that τ is a *NED* of μ^ρ . ■

We now show that when an *LDGT* is atomless, the Lebesgue unit interval $(L, \mathcal{L}, \ell)$ plays the essential role to determine if a *NED* of the *LDGT* is symmetric or not. With the saturated property studied in Keisler-Sun (2009), we now give a characterization result of a symmetric *NED* in an atomless *LDGT*. This also extends Noguchi (2009, Theorem 2) where there is no formulation of traits and the common action set is uncountable.

Theorem 4. *The following two conditions are equivalent for any NED τ of an atomless LDGT μ^ρ : (i) The Lebesgue unit interval $(L, \mathcal{L}, \ell)$ realizes τ , (ii) τ is symmetric.*

Proof: We first prove that (ii) implies (i). Let τ be a symmetric *NED* of *LDGT* μ^ρ . Then there exists $h : T \times \mathcal{V}_{(A, T, \rho)} \longrightarrow A$ such that $\tau(\text{graph of } h) = 1$. For any *LSGT* $\mathcal{G} : L \longrightarrow T \times \mathcal{V}_{(A, T, \rho)}$ satisfying $\ell\mathcal{G}^{-1} = \mu^\rho$, let $f = h \circ \mathcal{G}$. If B is any Borel subset of $T \times \mathcal{V}_{(A, T, \rho)} \times A$ then $\tau(B) = \mu^\rho(\{(t, u) : (t, u, h(t, u)) \in B\}) = \ell(\{i : (\mathcal{G}(i), f(i)) \in B\})$. Here, $\tau = \ell(\mathcal{G}, f)^{-1}$. We will show that f is a Nash equilibrium of $\mathcal{G}$. Since τ is a *NED*, $\tau(B(\tau)) = \ell((\mathcal{G}, f)^{-1}(B(\tau))) = \ell(\{i \in L : (\mathcal{G}(i), f(i)) \in B(\tau)\}) = 1$. So, for almost all $i \in L$, $(\mathcal{G}(i), f(i)) \in B(\tau)$. It is also easy to verify that $\tau_{T \times A} = \ell(\mathcal{G}_1, f)^{-1}$. Thus, f is a Nash equilibrium of $\mathcal{G}$.

For the converse part. Now suppose for any *LSGT* $\mathcal{G} : L \longrightarrow T \times \mathcal{V}_{(A, T, \rho)}$ satisfying $\ell\mathcal{G}^{-1} = \mu^\rho$, there is a Nash equilibrium f of $\mathcal{G}$ such that $\ell(\mathcal{G}, f)^{-1} = \tau$. The fact that

$(L, \mathcal{L}, \ell)$ realizes τ simply means that $(L, \mathcal{L}, \ell)$ has the saturated property for $\ell(\mathcal{G}, f)^{-1}$. Moreover, because μ^ρ is atomless, by Keisler-Sun (2009, Proposition 2.4), we know that f is $\sigma(\mathcal{G})$ -measurable. Thus, there exists a measurable function $h : T \times \mathcal{V}_{(A, T, \rho)} \longrightarrow A$ satisfying $f = h \circ \mathcal{G}$ by Aliprantis-Border (2007, Theorem 4.41). Denote the range of $\mathcal{G}$ by $R(\mathcal{G})$. We first show that $\tau(\text{graph of } h) = 1$.

$$\begin{aligned} \tau(\text{graph of } h) &= \tau(\{(t, v, h(t, v)) : (t, v) \in R(\mathcal{G})\}) + \tau(\{(t, v, h(t, v)) : (t, v) \notin R(\mathcal{G})\}) \\ &\geq \tau(\{(t, v, h(t, v)) : (t, v) \in R(\mathcal{G})\}) = \tau(\{(\mathcal{G}(i), f(i)) : i \in L\}) = \ell(L) = 1. \end{aligned}$$

As $\tau = \ell(\mathcal{G}, f)^{-1}$, it is clear that $\tau_{T \times A} = \ell(\mathcal{G}_1, f)^{-1}$. Since f is a Nash equilibrium, graph of $h \subseteq B(\tau)$ and $\tau(B(\tau)) = 1$. Therefore, τ is symmetrized. $\blacksquare$

Our last result furnishes a necessary and sufficient condition for the situation when a *NED* of an atomless *LDGT* cannot be realized by the Lebesgue unit interval.

Corollary 1. *An atomless probability space $(I, \mathcal{I}, \lambda)$ realizes a non-symmetric *NED* of an atomless *LDGT* if and only if it is saturated.*

Proof: It follows from Proposition 2 that a saturated space can realize any *NED* for any *LDGT*. We only need to show the sufficient part. Let τ be a non-symmetric *NED* of an atomless *LDGT*. By Theorem 4, the Lebesgue unit interval $(L, \mathcal{L}, \ell)$ cannot realize τ . That is to say, the Lebesgue unit interval does not have the saturation property for τ . However, $(I, \mathcal{I}, \lambda)$ has the saturation property for τ . It follows from Keisler-Sun (2009, Theorem 2.7) that $(I, \mathcal{I}, \lambda)$ must be saturated. $\blacksquare$

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